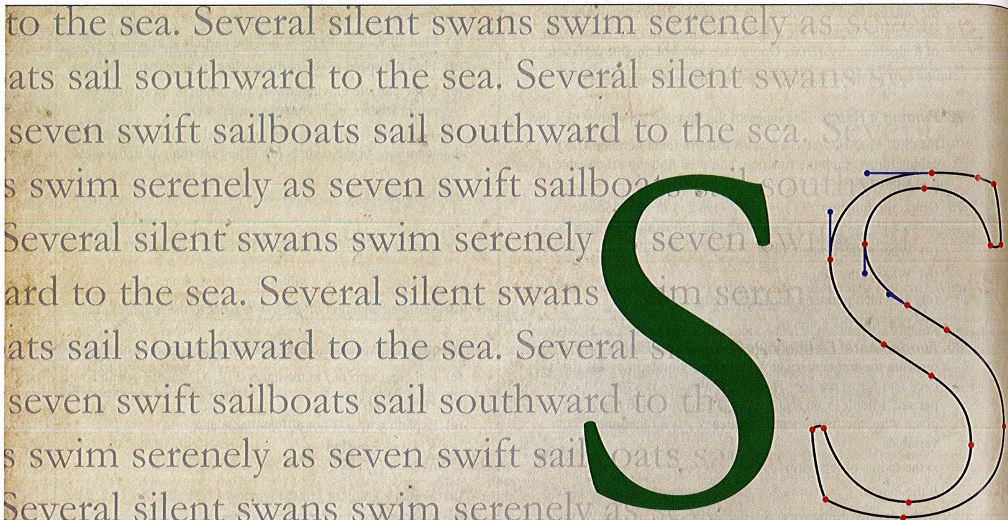
CHAPTER

4 More Derivatives



The letter S above is pieced together from curves joined at the red dots. Each piece of the outline is defined by parametric equations $x = f(t)$ and $y = g(t)$, where f and g are cubic polynomials and the parameter t runs from 0 to 1. In a font design program such as Fontlab™ you can drag the control points (blue dots) to change the shape of each piece of the outline. (We show control points for just two pieces of the outline.) As you drag the control points, you are actually changing the coefficients of the polynomials f and g. Using the techniques of Example 6 in Section 4.1 you can show that as long as a joint (or *node*) lies on the line joining its adjacent control points, the two curves that meet at the node will have the same slope there, making a smooth transition. When you send the letter to an output device like a printer, it uses the parametric equations to render the outline at the highest possible resolution.

4.1 Chain Rule

What you will learn about . . .

- Derivative of a Composite Function
- "Outside-Inside" Rule
- Repeated Use of the Chain Rule
- Slopes of Parametrized Curves
- Power Chain Rule

and why . . .

The Chain Rule is the most widely used differentiation rule in mathematics.

Derivative of a Composite Function

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is with the Chain Rule, which is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it.

EXAMPLE 1 Relating Derivatives

The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

SOLUTION

We have

$$\frac{dy}{dx} = 6, \quad \frac{dy}{du} = 2, \quad \frac{du}{dx} = 3.$$

Since $6 = 2 \cdot 3$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Now Try Exercise 1.

Is it an accident that $dy/dx = dy/du \cdot du/dx$?

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For $y = f(u)$ and $u = g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x. This is much like the effect of a multiple gear train (Figure 4.1).

Let us try again on another function.

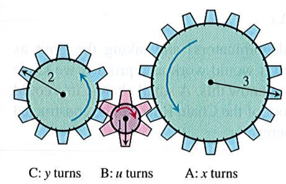


Figure 4.1 When gear A makes x turns, gear B makes u turns, and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ and $u = 3x$, so $y = 3x/2$. Thus $dy/du = 1/2$, $du/dx = 3$, and $dy/dx = 3/2 = (dy/du)(du/dx)$.

EXAMPLE 2 Relating Derivatives

The polynomial $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x.\end{aligned}$$

Also,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x.\end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}.$$

Now Try Exercise 5.

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x (Figure 4.2). This is known as the Chain Rule.

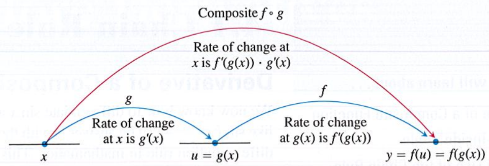


Figure 4.2 Rates of change multiply: the derivative of $f \circ g$ at x is the derivative of f at the point $g(x)$ times the derivative of g at x.

RULE 8 The Chain Rule

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

It would be tempting to try to prove the Chain Rule by writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

(a true statement about fractions with nonzero denominators) and taking the limit as $\Delta x \to 0$. This is essentially what is happening, and it would work as a proof if we knew that Δu, the change in u, was nonzero; but we do not know this. A small change in x could conceivably produce no change in u. An air-tight proof of the Chain Rule can be constructed through a different approach, but we will omit it here.

EXAMPLE 3 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

SOLUTION

We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t \sin(t^2 + 1). \end{aligned}$$

Now Try Exercise 9.

"Outside-Inside" Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

EXAMPLE 4 Differentiating from the Outside In

Differentiate $\sin(x^2 + x)$ with respect to x.

SOLUTION

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{\text{inside}} = \cos\underbrace{(x^2 + x)}_{\text{inside left alone}} \cdot \underbrace{(2x + 1)}_{\text{derivative of the inside}}$$

Now Try Exercise 13.

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example:

EXAMPLE 5 A Three-Link "Chain"

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

SOLUTION

Notice here that tan is a function of $5 - \sin 2t$, while sin is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$

Now Try Exercise 23.

Slopes of Parametrized Curves

A parametrized curve $(x(t), y(t))$ is *differentiable at t* if x and y are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Finding dy/dx Parametrically

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{1}$$

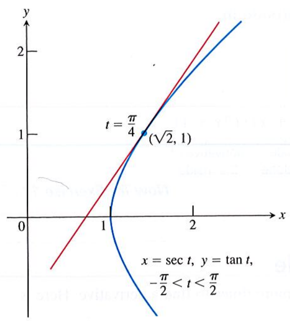


Figure 4.3 The hyperbola branch in Example 6. Equation 1 applies for every point on the graph except $(1, 0)$. Can you state why Equation 1 fails at $(1, 0)$?

EXAMPLE 6 Differentiating with a Parameter

Find the line tangent to the right-hand hyperbola branch defined parametrically by

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$ (Figure 4.3).

SOLUTION

All three of the derivatives in Equation 1 exist and $dx/dt = \sec t \tan t \neq 0$ at the indicated point. Therefore, Equation 1 applies and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{\sec^2 t}{\sec t \tan t} \\ &= \frac{\sec t}{\tan t} \\ &= \csc t. \end{aligned}$$

Setting $t = \pi/4$ gives

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \csc(\pi/4) = \sqrt{2}.$$

The equation of the tangent line is

$$\begin{aligned} y - 1 &= \sqrt{2}(x - \sqrt{2}) \\ y &= \sqrt{2}x - 2 + 1 \\ y &= \sqrt{2}x - 1. \end{aligned}$$

Now Try Exercise 41.

Power Chain Rule

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}.$$

Here's an example of how it works: If n is an integer and $f(u) = u^n$, the Power Rules (Rules 2 and 7) tell us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the Power Chain Rule:

$$\frac{d}{dx} u^n = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 7 Finding Slope

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

SOLUTION

(a)
$$\begin{aligned} \frac{dy}{dx} &= 5\sin^4 x \cdot \frac{d}{dx}\sin x \qquad \text{Power Chain Rule with } u = \sin x,\ n = 5 \\ &= 5\sin^4 x \cos x \end{aligned}$$

The tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}.$$

(b)
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(1 - 2x)^{-3} \\ &= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) \qquad \text{Power Chain Rule with } u = (1 - 2x),\ n = -3 \\ &= -3(1 - 2x)^{-4} \cdot (-2) \\ &= \frac{6}{(1 - 2x)^4} \end{aligned}$$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

the quotient of two positive numbers. *Now Try Exercise 53.*

EXAMPLE 8 Radians Versus Degrees

It is important to remember that the formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians. By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Figure 4.4.

The factor $\pi/180$, annoying in the first derivative, would compound with repeated differentiation. We see at a glance the compelling reason for the use of radian measure.

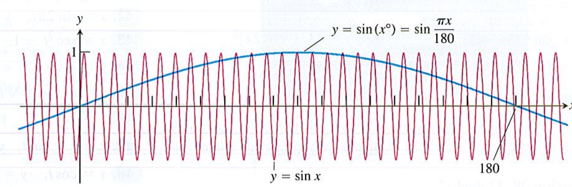


Figure 4.4 $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$. (Example 8)

Quick Review 4.1 (For help, go to Sections 1.2 and 1.6.)

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–5, let $f(x) = \sin x$, $g(x) = x^2 + 1$, and $h(x) = 7x$. Write a simplified expression for the composite function.

1. $f(g(x))$
2. $f(g(h(x)))$
3. $(g \circ h)(x)$
4. $(h \circ g)(x)$
5. $f\left(\dfrac{g(x)}{h(x)}\right)$

In Exercises 6–10, let $f(x) = \cos x$, $g(x) = \sqrt{x+2}$, and $h(x) = 3x^2$. Write the given function as a composite of two or more of f, g, and h. For example, $\cos 3x^2$ is $f(h(x))$.

6. $\sqrt{\cos x + 2}$
7. $\sqrt{3\cos^2 x + 2}$
8. $3\cos x + 6$
9. $\cos 27x^4$
10. $\cos\sqrt{2 + 3x^2}$

Section 4.1 Exercises

In Exercises 1–8, use the given substitution and the Chain Rule to find dy/dx.

1. $y = \sin(3x+1)$, $u = 3x + 1$
2. $y = \sin(7 - 5x)$, $u = 7 - 5x$
3. $y = \cos(\sqrt{3}x)$, $u = \sqrt{3}x$
4. $y = \tan(2x - x^3)$, $u = 2x - x^3$
5. $y = \left(\dfrac{\sin x}{1 + \cos x}\right)^2$, $u = \dfrac{\sin x}{1 + \cos x}$
6. $y = 5\cot\left(\dfrac{2}{x}\right)$, $u = \dfrac{2}{x}$
7. $y = \cos(\sin x)$, $u = \sin x$
8. $y = \sec(\tan x)$, $u = \tan x$

In Exercises 9–12, an object moves along the x-axis so that its position at any time $t \ge 0$ is given by $x(t) = s(t)$. Find the velocity of the object as a function of t.

9. $s = \cos\left(\dfrac{\pi}{2} - 3t\right)$
10. $s = t\cos(\pi - 4t)$
11. $s = \dfrac{4}{3\pi}\sin 3t + \dfrac{4}{5\pi}\cos 5t$
12. $s = \sin\left(\dfrac{3\pi}{2}t\right) + \cos\left(\dfrac{7\pi}{4}t\right)$

In Exercises 13–24, find dy/dx. If you are unsure of your answer, use NDER to support your computation.

13. $y = (x + \sqrt{x})^{-2}$
14. $y = (\csc x + \cot x)^{-1}$
15. $y = \sin^{-5} x - \cos^3 x$
16. $y = x^3(2x - 5)^4$
17. $y = \sin^3 x \tan 4x$
18. $y = 4\sqrt{\sec x + \tan x}$
19. $y = \dfrac{3}{\sqrt{2x + 1}}$
20. $y = \dfrac{x}{\sqrt{1 + x^2}}$
21. $y = \sin^2(3x - 2)$
22. $y = (1 + \cos 2x)^2$
23. $y = (1 + \cos^2 7x)^3$
24. $y = \sqrt{\tan 5x}$

In Exercises 25–28 find $dr/d\theta$.

25. $r = \tan(2 - \theta)$
26. $r = \sec 2\theta \tan 2\theta$
27. $r = \sqrt{\theta \sin\theta}$
28. $r = 2\theta\sqrt{\sec\theta}$

In Exercises 29–32, find y''.

29. $y = \tan x$
30. $y = \cot x$
31. $y = \cot(3x - 1)$
32. $y = 9\tan(x/3)$

In Exercises 33–38, find the value of $(f \circ g)'$ at the given value of x.

33. $f(u) = u^5 + 1$, $u = g(x) = \sqrt{x}$, $x = 1$
34. $f(u) = 1 - \dfrac{1}{u}$, $u = g(x) = \dfrac{1}{1 - x}$, $x = -1$
35. $f(u) = \cot\dfrac{\pi u}{10}$, $u = g(x) = 5\sqrt{x}$, $x = 1$
36. $f(u) = u + \dfrac{1}{\cos^2 u}$, $u = g(x) = \pi x$, $x = \dfrac{1}{4}$
37. $f(u) = \dfrac{2u}{u^2 + 1}$, $u = g(x) = 10x^2 + x + 1$, $x = 0$
38. $f(u) = \left(\dfrac{u - 1}{u + 1}\right)^2$, $u = g(x) = \dfrac{1}{x^2} - 1$, $x = -1$

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 39 and 40.

39. Find dy/dx if $y = \cos(6x + 2)$ by writing y as a composite with
(a) $y = \cos u$ and $u = 6x + 2$.
(b) $y = \cos 2u$ and $u = 3x + 1$.

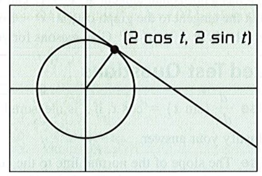
40. Find dy/dx if $y = \sin(x^2 + 1)$ by writing y as a composite with
(a) $y = \sin(u + 1)$ and $u = x^2$.
(b) $y = \sin u$ and $u = x^2 + 1$.

In Exercises 41–48, find the equation of the line tangent to the curve at the point defined by the given value of t.

41. $x = 2\cos t$, $y = 2\sin t$, $t = \pi/4$
42. $x = \sin 2\pi t$, $y = \cos 2\pi t$, $t = -1/6$
43. $x = \sec^2 t - 1$, $y = \tan t$, $t = -\pi/4$
44. $x = \sec t$, $y = \tan t$, $t = \pi/6$
45. $x = t$, $y = \sqrt{t}$, $t = 1/4$
46. $x = 2t^2 + 3$, $y = t^4$, $t = -1$
47. $x = t - \sin t$, $y = 1 - \cos t$, $t = \pi/3$
48. $x = \cos t$, $y = 1 + \sin t$, $t = \pi/2$

49. Let $x = t^2 + t$, and let $y = \sin t$.
(a) Find dy/dx as a function of t.
(b) Find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$ as a function of t.
(c) Find $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ as a function of t. Use the Chain Rule and your answer from part (b).
(d) Which of the expressions in parts (b) and (c) is d^2y/dx^2?

50. A circle of radius 2 and center $(0, 0)$ can be parametrized by the equations $x = 2\cos t$ and $y = 2\sin t$. Show that for any value of t, the line tangent to the circle at $(2\cos t, 2\sin t)$ is perpendicular to the radius.



51. Let $s = \cos\theta$. Evaluate ds/dt when $\theta = 3\pi/2$ and $d\theta/dt = 5$.

52. Let $y = x^2 + 7x - 5$. Evaluate dy/dt when $x = 1$ and $dx/dt = 1/3$.

53. What is the largest value possible for the slope of the curve $y = \sin(x/2)$?

54. Write an equation for the tangent to the curve $y = \sin mx$ at the origin.

55. Find the lines that are tangent and normal to the curve $y = 2\tan(\pi x/4)$ at $x = 1$. Support your answer graphically.

56. ***Working with Numerical Values*** Suppose that functions f and g and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Evaluate the derivatives with respect to x of the following combinations at the given value of x.
(a) $2f(x)$ at $x = 2$
(b) $f(x) + g(x)$ at $x = 3$
(c) $f(x)\cdot g(x)$ at $x = 3$
(d) $f(x)/g(x)$ at $x = 2$
(e) $f(g(x))$ at $x = 2$
(f) $\sqrt{f(x)}$ at $x = 2$
(g) $1/g^2(x)$ at $x = 3$
(h) $\sqrt{f^2(x) + g^2(x)}$ at $x = 2$

57. ***Extension of Example 8*** Show that $\dfrac{d}{dx}\cos(x°)$ is $-\dfrac{\pi}{180}\sin(x°)$.

58. ***Working with Numerical Values*** Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Evaluate the derivatives with respect to x of the following combinations at the given value of x.
(a) $5f(x) - g(x)$, $x = 1$
(b) $f(x)g^3(x)$, $x = 0$
(c) $\dfrac{f(x)}{g(x) + 1}$, $x = 1$
(d) $f(g(x))$, $x = 0$
(e) $g(f(x))$, $x = 0$
(f) $(g(x) + f(x))^{-2}$, $x = 1$
(g) $f(x + g(x))$, $x = 0$

59. ***Orthogonal Curves*** Two curves are said to cross at right angles if their tangents are perpendicular at the crossing point. The technical word for "crossing at right angles" is **orthogonal.** Show that the curves $y = \sin 2x$ and $y = -\sin(x/2)$ are orthogonal at the origin. Draw both graphs and both tangents in a square viewing window.

60. **Writing to Learn** Explain why the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

is not simply the well-known rule for multiplying fractions.

61. ***Running Machinery Too Fast*** Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

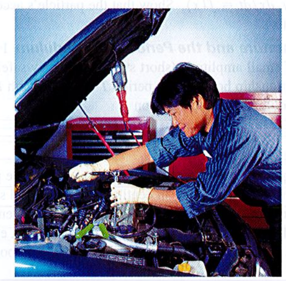


Figure 4.5 The internal forces in the engine get so large that they tear the engine apart when the velocity is too great.

62. Group Activity *Temperatures in Fairbanks, Alaska.* The graph in Figure 4.6 shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25.$$

(a) On what day is the temperature increasing the fastest?

(b) About how many degrees per day is the temperature increasing when it is increasing at its fastest?

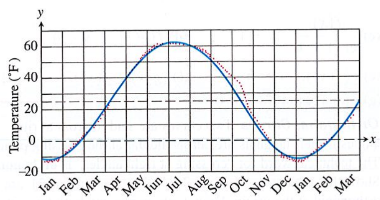


Figure 4.6 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points, and the approximating sine function (Exercise 62).

63. *Particle Motion* The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

64. *Constant Acceleration* Suppose the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.

65. *Falling Meteorite* The velocity of a heavy meteorite entering the earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s kilometers from the earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

66. *Particle Acceleration* A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

67. *Temperature and the Period of a Pendulum* For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

68. Writing to Learn *Chain Rule* Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad\text{and}\quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

69. *Tangents* Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

Standardized Test Questions

70. True or False $\frac{d}{dx}(\sin x) = \cos x$, if x is measured in degrees or radians. Justify your answer.

71. True or False The slope of the normal line to the curve $x = 3\cos t$, $y = 3\sin t$ at $t = \pi/4$ is -1. Justify your answer.

72. Multiple Choice Which of the following is dy/dx if $y = \tan(4x)$?

(A) $4\sec(4x)\tan(4x)$ (B) $\sec(4x)\tan(4x)$ (C) $4\cot(4x)$

(D) $\sec^2(4x)$ (E) $4\sec^2(4x)$

73. Multiple Choice Which of the following is dy/dx if $y = \cos^2(x^3 + x^2)$?

(A) $-2(3x^2 + 2x)$

(B) $-(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(C) $-2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(D) $2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(E) $2(3x^2 + 2x)$

In Exercises 74 and 75, use the curve defined by the parametric equations $x = t - \cos t$, $y = -1 + \sin t$.

74. Multiple Choice Which of the following is an equation of the tangent line to the curve at $t = 0$?

(A) $y = x$ (B) $y = -x$ (C) $y = x + 2$

(D) $y = x - 2$ (E) $y = -x - 2$

75. Multiple Choice At which of the following values of t is $dy/dx = 0$?

(A) $t = \pi/4$ (B) $t = \pi/2$ (C) $t = 3\pi/4$

(D) $t = \pi$ (E) $t = 2\pi$

Explorations

76. *The Derivative of sin 2x* Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x + h) - \sin 2x}{h}$$

for $h = 1.0$, 0.5, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

77. *The Derivative of cos (x^2)* Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on screen, graph

$$y = \frac{\cos[(x + h)^2] - \cos(x)^2}{h}$$

for $h = 1.0$, 0.7, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

Extending the Ideas

78. *Absolute Value Functions* Let u be a differentiable function of x.

(a) Show that $\frac{d}{dx}|u| = u'\frac{u}{|u|}$.

(b) Use part (a) to find the derivatives of $f(x) = |x^2 - 9|$ and $g(x) = |x|\sin x$.

79. *Geometric and Arithmetic Mean* The geometric mean of u and v is $G = \sqrt{uv}$ and the arithmetic mean is $A = (u + v)/2$. Show that if $u = x$, $v = x + c$, c a real number, then

$$\frac{dG}{dx} = \frac{A}{G}.$$